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Transient Bohm–Gross waves radiated by a point source on the basis of the Klein–Gordon equation

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Abstract. A formalism based on the theory of distributions associated with the Borel or Poisson's transformations is developed for obtaining fundamental solutions for point sources immersed in a hydrodynamical warm plasma. The method allows an analytical solution of the linear Klein–Gordon differential equation. Algebraic solutions of electrostatic radiation problems, expressed in term of two-variable Lommel functions are proposed.

1. Introduction

The Klein–Gordon equation is of great interest in physics. Being the simplest of the energy-preserving dispersive hyperbolic differential equations, it serves as a useful mathematical model for several physical phenomena. For example, the solution of this equation and some related forms can be used to describe the vertical propagation of waves in a stratified atmosphere, waves on strings with elastic supports, waves in magnetic flux tubes, Alfvén waves under certain propagation conditions and electromagnetic waves in cold plasma and in plane parallel wave guides.

Related to the problem of acoustic gravity wave propagation in the atmosphere, Robinson [1] has recently reported some characteristics of what are termed algebraic solutions for the Klein–Gordon equation. These solutions involve special functions and they are referred to as algebraic in order to distinguish them from the more familiar harmonic or modal solutions. The impulsive excitation of longitudinal waves by an external point source immersed within an isotropic warm plasma has been re-examined more recently [2]. It is true that this mode of the plasma satisfying the Bohm–Gross dispersion relationship [3] may be described by the Klein–Gordon equation. However, for the electric wave field to behave curl-free, it is necessary that the expected potential response is governed by Poisson's equation. The simplest propagation equation including the above-mentioned characteristics should be a higher-order partial differential equation which contains both Laplacian and Klein–Gordon operators. Starting from such an equation, the response of the plasma may be expressed in terms of Lommel functions of two real variables [4]. This exact algebraic expression can be regarded as an infinite sum of the Robinson-type solutions.

This paper is concerned with obtaining previous results on an impulsive excitation of the Bohm–Gross waves [2]. Later, the analysis is extended to the problem of electrostatic wave generation by a causal and forced pulsatory point charge of semi-infinite duration, which may be used to exemplify a physically more realistic situation than the idealized impulsive

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excitation. The Laplace transform technique is much in favour among plasma physicists and others as a tool for obtaining the algebraic solution for such transient excitations, but here, in the light of Robinson's work [1], an alternative approach based on the theory of distributions (or generalized functions) is proposed. As a consequence, our main interest here turns out to be in a detailed description of the computational procedure. The investigation of the physical problem under consideration is used to highlight the main aspects underlying the relevance of the formalism.

One of the key points of the technique consists of an exact expansion of the driving term of the nonhomogeneous evolution equation as a series of the generalized time derivatives of the impulse distribution. For a wide class of real-valued and infinitely differentiable functions, such an expansion can be obtained by means of the so-called Poisson transform method [5, 6]. The solution which is obtained from the linear superposition procedure is then expressed as a summation of successive distributional derivatives of the Green function (fundamental solution) of the Klein–Gordon equation. The necessary condition for this approach to be valid depends upon the convergence of the resulting infinite series. If, in some situations, such a condition is not fulfilled, the formalism can be modified to deal with the difficulty in the following way. Expanding the source term as a Taylor series about the origin of the time or space variable, the solution is shown to be a series of multiple iterated smoothing integrals of the Green function.

In the second part of the paper, basic assumptions and the equations governing uncoupled cold and warm plasma are presented. The method of solution which refines and generalizes the previous formalism of obtaining the response of a system governed by a nonhomogeneous linear Klein–Gordon equation is developed in section 3. As a first application, the algebraic solution of the problem of an impulsive delta function in time excitation (Green's function) of the Bohm–Gross wave is then recalled in section 4. Sections 5 and 6 deal with the switch-on harmonic in time excitation problem in the cases of dilute and dense plasma, respectively. Summary and concluding remarks close the paper in section 7.

2. Set of propagation equations

For the sake of simplicity, an electron hydrodynamical, isotropic and infinite plasma which is maintained globally neutral by the presence of a background of fixed positive ions is considered. The excited waves are supposed to be longitudinal, so the system may be characterized by a self-consistent set of hydrodynamical relations (i.e. conservation of mass and conservation of momentum) and Poisson's equation. The time derivative of the continuity equation and the force equation can be combined to yield a governing equation for the density fluctuations (see e.g., [7, 8]). Then, eliminating the density by use of Poisson's equation, we can show that the electric wave field potential $\phi(\boldsymbol{r}, t)$ satisfies the generalized Klein–Gordon differential equation given by

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 - c_T^2 \nabla^2\right) \nabla^2 \phi(\boldsymbol{r}, t) = \left(\frac{\partial^2}{\partial t^2} - c_T^2 \nabla^2\right) \nabla^2 \phi_{ext}(\boldsymbol{r}, t) \tag{2.1}$$

where ω_p denotes the plasma frequency, $c_T^2 = (3V_T^2/2)$ specifies the thermal speed, V_T, of the electronic fluid, and $\phi_{ext}(\boldsymbol{r}, t)$, the externally applied electric potential. Setting $\chi(\boldsymbol{r}, t) = [\phi(\boldsymbol{r}, t) - \phi_{ext}(\boldsymbol{r}, t)]$ as a new dependent variable in (2.1) and rearranging, we may rewrite this equation as

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 - c_T^2 \nabla^2\right) \nabla^2 \chi(\boldsymbol{r}, t) = \frac{\omega_p^2}{\varepsilon_o} Q_{ext}(\boldsymbol{r}, t). \tag{2.2}$$

Here, the external charge distribution $Q_{ext}(\boldsymbol{r}, t)$, related to the external potential and the free space permittivity ε_o has been introduced in (2.2) through Poisson's equation, $\nabla^2\phi_{ext} + Q_{ext}/\varepsilon_o = 0$. Also, it will be assumed that for $t < 0$, the external potential $\phi_{ext} \equiv 0$, hence the perturbations of the plasma are 'causal' and disappear in this region. This assumption destroys the reversible character of the present non-dissipative description. Also, we consider a punctual charge distribution source localized at the origin of the space coordinates, thus $Q_{ext}(\boldsymbol{r}, t)$ may be expressed as the product of $\delta_3(\boldsymbol{r})$, the three-dimensional Dirac delta function and an arbitrary function of time.

As in conventional problems, it is convenient to deal with uncoupled simplest systems rather than the composite Laplacian and Klein–Gordon differential operators on the left-hand side of (2.2). The separation of the problem is expected since in the case of a steady-state sinusoidal approximation, Rooy *et al* [9] showed that the response of the plasma, subject to an external forcing charge, can be represented by means of a superposition of the cold plasma oscillations and Bohm–Gross (i.e. warm plasma) wave. Defining the dependent variable as a summation of two terms, $\chi(\boldsymbol{r}, t) = \chi_C(\boldsymbol{r}, t) + \phi_{BG}(\boldsymbol{r}, t)$, equation (2.2) is equivalent to the set of partial differential equations,

$$\nabla^2\chi_C(\boldsymbol{r}, t) = S(\boldsymbol{r}, t) \tag{2.3}$$

and

$$\left(-\nabla^2 + \frac{1}{c_T^2}\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{c_T^2}\right)\phi_{BG}(r, t) = S(\boldsymbol{r}, t) \tag{2.4}$$

where the free term takes the form

$$S(\boldsymbol{r}, t) = (\omega_p/\varepsilon_o)H(t)\sin(\omega_p t) *_t Q_{ext}(\boldsymbol{r}, t) \tag{2.5}$$

and where the symbol $*_t$ is the operation of convolution with respect to time, and H stands for the unit step function of Heaviside. This separation into a set of equations (2.3) and (2.4) is most easily demonstrated with the aid of the commonly used Laplace transform method (see [2]). Notice that the term $H(t)\sin(\omega_p t)/\omega_p$ on the right-hand side of (2.5) is the causal Green function related to the linear differential operator $(\partial^2/\partial t^2 + \omega_p^2)$, hence the convolution operation turns out to be symbolically equivalent to the application of the inverse of this operator. The total solution $\chi(\boldsymbol{r}, t)$ has the unphysical property that the information propagates at arbitrary high speed. This instantaneous response of the plasma stems from the quasistatic approximation used in the present description.

A distinction must be drawn in the following sections between types of external charges. The first type, a finite-amplitude impulse, is exemplified by a delta function charge in time and space

$$Q_{ext}^{[\delta]}(\boldsymbol{r}, t) = q_o\delta_3(\boldsymbol{r})\delta(t). \tag{2.6}$$

Here q_o is a constant, and δ denotes the single variable Dirac delta function. The second type of charge, one which is switched on and maintained quasi-indefinitely, is exemplified by the switch-on of continuous oscillatory excitation

$$Q_{ext}^{[c]}(\boldsymbol{r}, t) = q_o\delta_3(\boldsymbol{r})H(t)\sin\omega_o t \tag{2.7}$$

where ω_o is the transmitter angular frequency. In the present treatment, we are concerned with both dilute, $\omega_o \geqslant \omega_p$, and overdense plasma, $\omega_o \leqslant \omega_p$.

3. Method of solution

First, we examine the perturbation related to the Bohm–Gross longitudinal wave. This response is the solution of the nonhomogeneous differential equation (2.4). Cauchy data prescribed at $t = 0$, will be here $\phi_{BG}(\boldsymbol{r}, 0) = (\partial/\partial t)\phi_{BG}(\boldsymbol{r}, 0) = 0$. If we consider a spherically symmetrical solution and opt for a new dependent variable $\psi(r,t) = r\phi_{BG}(\boldsymbol{r}, t)$, then (2.4) reduces to a partial differential equation in one space variable:

$$\mathcal{L}\psi(r,t) \equiv \left(-\frac{\partial^2}{\partial r^2} + \frac{1}{c_T^2}\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{c_T^2}\right)\psi(r,t) = rS(\boldsymbol{r}, t). \tag{3.1}$$

In (3.1) and in what follows, the symbol $\mathcal{L}$ denotes the one-dimensional linear Klein–Gordon operator, and r stands for the modulus of the position vector $\boldsymbol{r}$.

Robinson [1] discussed the general solution of the homogeneous Klein–Gordon equation, $\mathcal{L}g(r,t) = 0$. It has been shown that an allowable arbitrary solution to this partial differential equation may be expressed as an infinite series given by

$$g(r,t) = \sum_{n=-\infty}^{\infty} \alpha_n g_n(r,t)$$

with

$$g_n(r,t) = -c_T H(t - r/c_T)\left(\frac{t - r/c_T}{t + r/c_T}\right)^{n/2} J_n(\omega_p(t^2 - r^2/{c_T}^2)^{1/2}) \tag{3.2}$$

the α_n being arbitrary coefficients and J_n, the cylindrical Bessel function of order n. Noticing that for the particular case, $n = 0$, the function $g_0(r,t)$ constitutes simultaneously the classical and fundamental solution to the homogeneous and the associated inhomogeneous Green function equation, respectively, this author deduced (see appendix A) that for the equation

$$\mathcal{L}\varphi(r,t) = f(r,t) = \delta^{(j)}(r)\delta^{(k)}(t) \tag{3.3}$$

where $\delta^{(i)}(u)$ is the ith generalized derivative with respect to the argument of δ, the solution is

$$\varphi(r,t) = \frac{\partial^{j+k} g_0(r,t)}{\partial r^j \partial t^k} \tag{3.4}$$

where

$$g_0(r,t) = -c_T H(t - r/c_T) J_0(\omega_p(t^2 - r^2/{c_T}^2)^{1/2}). \tag{3.5}$$

Stated in this form, the outlined prescription applies to our problem if the free term on the right-hand side of (3.1) can be expanded into a linear combination of derivatives of the basic impulse function. Green and Messel [10] considered the expansion of any suitable function in a series of higher-order impulse distribution functions. The coefficients of the expansion are known to be related to the moments of the function under consideration. Subsequently, the method used by these authors appeared in the literature as the technique of Poisson's transform [5, 11, 12], some fundamental definitions and properties of which have been discussed by Fairman and Piovoso [13]. Note also that theorems relating the Poisson and Laplace transforms of functions are well established [6]. However, one of the simplest way to derive such series expansions is to notice that any function of time may be thought as a convolution between δ and the function itself. Then the algorithm commonly used for the evaluation of a convolution integral, proposed, for example, in [14], can be applied. The desired result consists then of a series in terms of the derivatives of the Dirac delta function and the moments of the transformed function. Because of the linear character of the description at hand, the superposition theorem applies, and the solution of (3.1) is deduced from the related infinite series of higher derivatives of the fundamental solution of $\mathcal{L}$ as

$$\psi(r,t) = \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{n!}\frac{\partial^n g_0(r,t)}{\partial t^n} \tag{3.6}$$

with

$$a_n = \int_0^\infty t_1^n [rS(\boldsymbol{r}, t_1)]\, \mathrm{d}t_1$$

being the nth moment of the source term.

According to the definition in (3.5), it is clear that $g_0(r, t)$ is a generalized function, and it has a jump or simple discontinuity at $t = (r/c_T)$. The differentiation operation used in (3.4) is understood as the distributional derivative. It is worth noting that, without regard to the discontinuity, the general relationship between $g_n(r, t)$ in (3.2) and all its derivatives is now well established [1]. Infinite series of g_n functions are then expected to represent the solution of the considered problems.

One can extend the method to negative integer powers $-j$ of the delta function on the right-hand side of (3.3). The function $\delta^{(-j)}$, $j \geqslant 1$, is then a j-fold repeated integral of δ, and by a similar approach as in (3.4), the solution becomes a repeated integral of $g_0(r, t)$. In order to find the linear response of the system, we develop the source term, $S(\boldsymbol{r}, t)$, in a power series of one of its arguments (say t). Then, the Klein–Gordon fundamental solution is convolved with each term of the resulting series. The Dirichlet's integral formula [15, 16]

$$\frac{1}{(n-1)!}\int_0^t f(t_1)(t-t_1)^{n-1}\, \mathrm{d}t_1 = \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} f(t_1)\, \mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_n \tag{3.7}$$

is applied to transform the time development of the solution. At once, the response of the plasma has the following series representation:

$$\psi(r, t) = \sum_{n=0}^{\infty} b_n \int_0^t \int_0^{t_{n+1}} \ldots \int_0^{t_2} g_0(r, t_1)\, \mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_{n+1} \tag{3.8}$$

with $b_n = [\partial^n (rS(\boldsymbol{r}, t))/\partial t^n]_{t=0}$, the nth order derivative of the source term evaluated at $t = 0$. In due course, we shall address the problem of how to deal with the multiple iterated integrals of the fundamental solution $g_0(r, t)$.

Note that if an arbitrary function $f(t)$ has the Taylor series expansion $f(t) = \sum_{n=0}^{\infty}(b_n/n!)t^n$, then one can define the related series $(\mathcal{B}f)(\tau) = \sum_{n=0}^{\infty} b_n \tau^n$. The function $\mathcal{B}f$ is the Borel transform of f. A relatively wide class of space and time variation function may be expressed as either Borel or Poisson's transformation series, but we must add that the necessary condition for the present method to be valid depends upon the convergence of the series (3.6) or (3.8). Here, the technique will be applied to the problem of the longitudinal warm plasma wave excitation. We shall see that, in order to find the form of the solution for different parameter values of the plasma, the present example gives an opportunity to explore the method in both the above-mentioned forms, i.e. (1) use of series of higher-order derivatives and (2) the multiple integrals series of the fundamental function.

4. Response to a temporal impulsive perturbation

4.1. Cold plasma response

The perturbation source is activated impulsively at time $t = 0$ within the plasma. The charge distribution takes the form $Q_{ext}^{[\delta]}(\boldsymbol{r}, t)$. Substituting relation (2.6) into (2.5) and Poisson's equation (2.3), we obtain the expressions for the externally applied electric potential and the source term of the coupled governing equations as

$$\phi_{ext}^{[\delta]}(\boldsymbol{r}, t) = \frac{q_o}{4\pi\varepsilon_o r}\delta(t) \tag{4.1}$$

$$S^{[\delta]}(\boldsymbol{r}, t) = \frac{q_o}{\varepsilon_o}\omega_p \delta_3(\boldsymbol{r}) H(t) \sin(\omega_p t). \tag{4.2}$$

Then, the solution of (2.3) with (4.2) may be calculated and the cold plasma response to the impulsive excitation is given by

$$\chi_C^{[\delta]}(\boldsymbol{r}, t) = -\frac{q_o}{4\pi\varepsilon_o r}\omega_p H(t)\sin(\omega_p t). \tag{4.3}$$

We see from (4.3) that the disturbance consists of time oscillations which cover instantaneously the entire region of the plasma.

4.2. *Warm plasma longitudinal wave*

We now examine the perturbation related to the Bohm–Gross longitudinal wave using the formalism described in the previous section. First, it is obvious to show that

$$S^{[\delta]}(\boldsymbol{r}, t) = \frac{q_o}{4\pi\varepsilon_o}\frac{\delta(r)}{r^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_p^{2k}}\delta^{(2k)}(t). \tag{4.4}$$

In (4.4), the three-dimensional delta function has been transformed into a one-dimensional one using the spherical symmetry of the system. Moreover, by the identity, $r\delta'(r) = -\delta(r)$, the space variation of the source term of (4.4) reduces to a first derivative of the impulse function with respect to r. As a result, the algebraic solution of our problem will consist of an infinite series of the derivatives of $g_0(r, t)$, and we may write that $\psi^{[\delta]}(r, t) = (q_o/4\pi\varepsilon_o)\Psi^{[\delta]}(r, t)$, where the function $\Psi^{[\delta]}(r, t)$ takes the formal expression

$$\Psi^{[\delta]}(r, t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_p^{2k}}\left(\frac{\partial^{2k}h(r, t)}{\partial t^{2k}}\right) \tag{4.5}$$

with

$$h(r, t) = \left(\frac{\partial g_0(r, t)}{\partial r}\right). \tag{4.6}$$

It is clear that the reduced electric potential, $\Psi^{[\delta]}(r, t)$, may also be put as a series of Bessel functions or a two-variable Lommel function [2]. To find such an expression, we must calculate the successive derivatives of the fundamental solution, $g_0(r, t)$, of the inhomogeneous Klein–Gordon equation.

Both functions $g_0(r, t)$ and $h(r, t)$ are infinitely differentiable on the real t line except at the point (r/c_T), where only left- and right-hand derivatives exist. In what follows, we denote by $[\partial f/\partial t]$, $[\partial^2 f/\partial t^2]$, etc, the functions obtained by differentiating $f(r, t)$ without regard to the jump, and let the discontinuity in $[\partial^m f/\partial t^m]$ at the jump be $\Delta f^{(m)}$. Then we have [17]

$$h(r, t) = h_1(r, t) - \frac{1}{c_T}\delta\left(t - \frac{r}{c_T}\right)\Delta g_0^{(0)} \tag{4.7}$$

with the notation $h_1(r, t) = [\partial g_0/\partial r]$, and for $m \geqslant 1$,

$$\frac{\partial^m h_1}{\partial t^m} = \left[\frac{\partial^m h_1}{\partial t^m}\right] + \sum_{j=0}^{m-1}\Delta h_1^{(m-1-j)}\frac{\partial^j\delta(t - r/c_T)}{\partial t^j}. \tag{4.8}$$

We substitute (4.8) in (4.5) and rearrange the resulting double summation using the relation

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}u_{j,k-j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}u_{j,k}. \tag{4.9}$$

Then, making use of Poisson's transformation formulae

$$H(t)\sin(\omega_p t) = \frac{1}{\omega_p}\sum_{n=0}^{\infty}\frac{(-1)^n}{\omega_p^{2n}}\delta^{(2n)}(t) \tag{4.10}$$

and

$$H(t)\cos(\omega_p t) = \frac{1}{\omega_p^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\omega_p^{2n}}\delta^{(2n+1)}(t) \tag{4.11}$$

we obtain the following expression for the space and time variation of the electric potential defined by (4.5)

$$\Psi^{[\delta]}(r,t) = \mathcal{F}(r,t) - H(t - r/c_T)\{\mathcal{D}_1(r)\cos[\omega_p(t - r/c_T)] + \mathcal{D}_2(r)\sin[\omega_p(t - r/c_T)]\} \tag{4.12}$$

with

$$\mathcal{F}(r,t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_p^{2k}}\left[\frac{\partial^{2k}h_1}{\partial t^{2k}}\right] \tag{4.13}$$

$$\mathcal{D}_1(r) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_p^{2k}}\Delta h_1^{(2k)} \tag{4.14}$$

and

$$\mathcal{D}_2(r) = \frac{\omega_p}{c_T}\Delta g_0^{(0)} + \frac{1}{\omega_p}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_p^{2k}}\Delta h_1^{(2k+1)}. \tag{4.15}$$

In (4.12), we note that the last two terms which contain the functions $\mathcal{D}_1(r)$ and $\mathcal{D}_2(r)$, respectively, represent the proper contribution of the jump. They embody the effects of the discontinuity at $t = (r/c_T)$ of all the derivatives of the generalized function $g_0(r,t)$. The following equalities are easily established: $\mathcal{D}_1(r) = \delta(t - r/c_T)\mathcal{F}(r,t)$, and $\mathcal{D}_2(r) = -\omega_p + \omega_p^{-1}\delta(t - r/c_T)[\partial\mathcal{F}(r,t)/\partial t]$. Once $\mathcal{F}(r,t)$ is known, we can obtain $\mathcal{D}_1(r)$ and $\mathcal{D}_2(r)$.

To proceed further, we refer again to Robinson's note [1] which employs Schläfli's integral formula to write any function $g_n(r,t)$ defined in (3.2) in the form

$$g_n(r,t) = \frac{c_T}{\pi}H(t - r/c_T)\int_0^{\pi} e^{\omega_p(r/c_T)\cos\zeta}\cos(\omega_p t\sin\zeta - n\zeta)\,d\zeta. \tag{4.16}$$

This representation is particularly advantageous in certain aspects of the handling of g_n-functions since the time and space variables are separated in (4.16). This leads to simplification in obtaining their higher-order derivatives. In particular, if we set $n = 0$, by a successive differentiation under the integral sign in (4.16), and then interchange the operations of integration and summation with respect to k, we can show that

$$\mathcal{F}(r,t) = 2\omega_p H(t - r/c_T)\frac{1}{\pi}\int_0^{\pi} e^{\omega_p(r/c_T)\cos\zeta}\frac{\cos\theta + \cos(\theta + 2\zeta)}{2 + 2\cos 2\zeta}\,d\zeta \tag{4.17}$$

where the notation, $\theta = (\omega_p t\sin\zeta - \zeta)$, has been used for convenience. Such an integral may then be converted back to an infinite series if we make use of the formula [18]

$$\frac{\cos\theta - \kappa\cos(\theta - 2\zeta)}{1 - 2\kappa\cos 2\zeta + \kappa^2} = \sum_{n=0}^{\infty}\kappa^n\cos(\theta + 2n\zeta). \tag{4.18}$$

Here, the parameter $\kappa = -1$. Using (4.18) in (4.17), a simple identification of each term of the series enables us to infer the following results for $\mathcal{F}(r,t)$, $\mathcal{D}_1(r)$, and $\mathcal{D}_2(r)$:

$$\mathcal{F}(r,t) = 2\omega_p H(t - r/c_T)\sum_{n=0}^{\infty}(-1)^n\left(\frac{t - r/c_T}{t + r/c_T}\right)^{-(n+\frac{1}{2})} J_{2n+1}(\omega_p(t^2 - r^2/c_T{}^2)^{1/2}). \tag{4.19}$$

$\mathcal{D}_1(r) = -2\omega_p \sin(\omega_p r/c_T)$, and $\mathcal{D}_2(r) = -2\omega_p \cos(\omega_p r/c_T)$. The contribution of the jump, as defined in (4.12), reduces at once to the simple trigonometric function, $2\omega_p H(t - r/c_T)\sin(\omega_p t)$. By virtue of the identity [4]

$$\sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{x}{y}\right)^{2n+1} J_{2n+1}(y) = \sin\left(\frac{x}{2} + \frac{y^2}{2x}\right)$$

we condense both the negative power series (4.19) and the sine function associated with the discontinuity of $g_0(r, t)$ as a single series. The relation (4.12) leads, therefore, to the expression

$$\Psi^{[\delta]}(\boldsymbol{r}, t) = \omega_p H(t - r/c_T) 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{t - r/c_T}{t + r/c_T}\right)^{n+\frac{1}{2}} J_{2n+1}(\omega_p (t^2 - r^2/c_T^2)^{1/2}). \tag{4.20}$$

In the same manner as in our previous paper [2], it turns out that the Bohm–Gross response to an impulsive excitation may be expressed quite simply in terms of a first-order Lommel function of two real variables as [4]

$$\phi_{GB}^{[\delta]}(\boldsymbol{r}, t) = \frac{q_o}{4\pi\varepsilon_o r} \omega_p H(t - r/c_T) 2 U_1(\omega_p (t - r/c_T), \omega_p (t^2 - r^2/c_T^2)^{1/2}). \tag{4.21}$$

There are occasions when it is advantageous to employ expression (4.21) rather than the equivalent series inferred from (4.20). First, for numerical computation, Buckley [19] used a direct summation of the Bessel functions series. However, the usefulness of this technique was somewhat limited since some tricks are involved in the practical calculation of the series, notably for large arguments [20], whereas some improved algorithms for computation of Lommel functions are now available ([2] and references cited therein). The second point is partially a consequence of the previous one, because local or asymptotic forms of the response may be readily deduced from the expression (4.21). They can be used effectively to gain some more understanding of the physical mechanism of the wave radiation and propagation. Results of such an insight into the Bohm–Gross wave dynamics were presented comprehensively in [21]. In this paper, however, the space-time variation of the amplitude of the reduced Bohm–Gross wave potential, $\omega_p^{-1}\Psi^{[\delta]}(\boldsymbol{r}, t)$ (see (4.20)) is given in figure 1. Here, the normalized time is defined by $\omega_p(t - r/c_T)$, whereas the normalized distance is defined by $\omega_p(r/c_T)$. The envelope of the generated signal, at a fixed position in space, consists of a monotonous rising curve whose maximum is localized at $t = +\infty$.

5. Response to a continuous excitation in a dilute plasma

We shall next consider a situation in which a sinusoidal excitation of semi-infinite duration is applied to the plasma. Obviously, the externally applied electrostatic potential can be written as

$$\phi_{ext}^{[\omega]}(\boldsymbol{r}, t) = \frac{q_o}{4\pi\varepsilon_o r} H(t) \sin(\omega_o t) \tag{5.1}$$

and the driving term of the governing equations (2.3) and (2.4) takes the form

$$S^{[\omega]}(\boldsymbol{r}, t) = \frac{q_o}{\varepsilon_o} \frac{\omega_p}{\omega_o^2 - \omega_p^2} \delta_3(\boldsymbol{r}) H(t) [\omega_o \sin(\omega_p t) - \omega_p \sin(\omega_o t)] \tag{5.2}$$

that corresponds to the convolution product (2.5), the external charge varying in time as (2.7).

Note that a pulsating source of finite time duration is the most likely kind of signal which occurs in practice. In such a situation, the response of the plasma may be deduced as a linear combination of the solution proposed here. This interpretation follows if we consider the time course of the source of finite duration T as a difference between two oscillatory unit step functions, one occurring at $t = 0$ and the other at $t = T$.

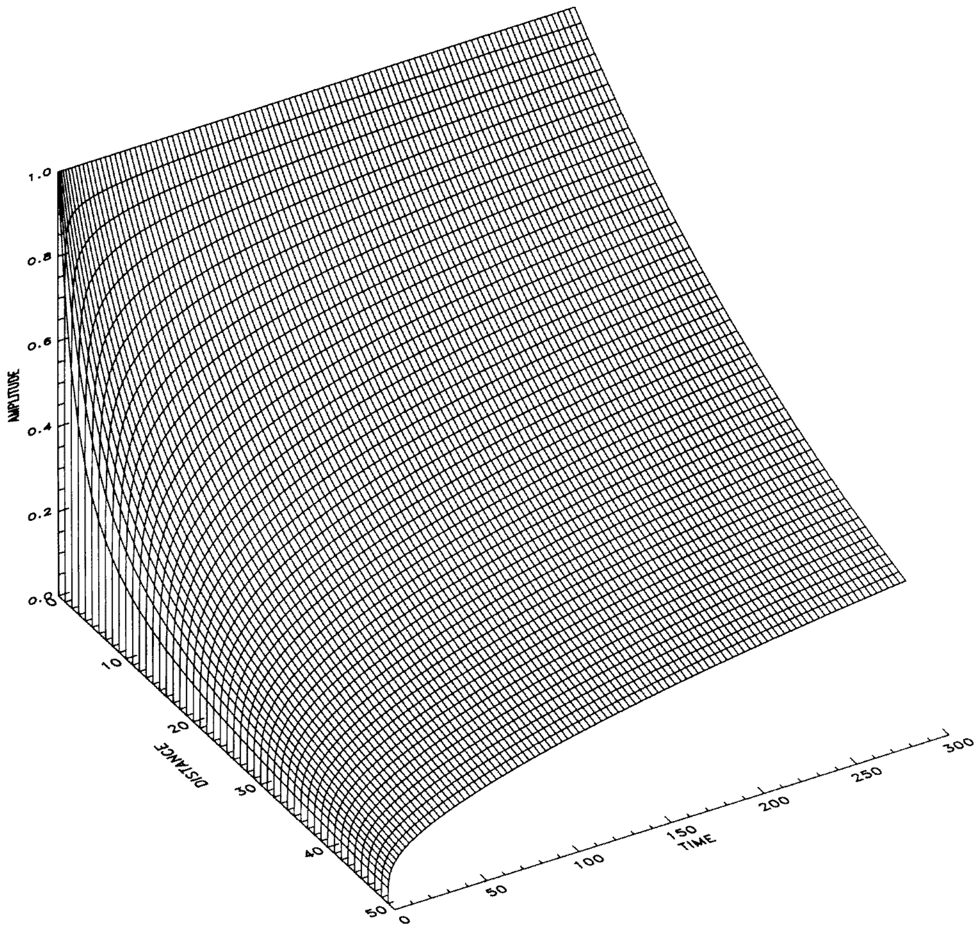

Figure 1. Space-time variation of the envelope curve of the function $\omega_p^{-1}\Psi^{[\delta]}(\boldsymbol{r}, t)$ (see (4.20)). The definition of the normalized time and the normalized distance is given in the text.

The cold plasma response may again be obtained by a straightforward integration of convolution of two simple sine functions. The result takes the form

$$\chi_C^{[\omega]}(\boldsymbol{r}, t) = \frac{q_o}{4\pi\varepsilon_o r} H(t)[P_o \sin(\omega_o t) - P_p \sin(\omega_p t)] \tag{5.3}$$

with

$$P_o = \frac{\omega_p^2}{\omega_o^2 - \omega_p^2} \qquad \text{and} \qquad P_p = \frac{\omega_o \omega_p}{\omega_o^2 - \omega_p^2}.$$

The response consists then of instantaneous beating oscillations of the external antenna vibrations and the cold plasma eigenmode.

In order to derive the Bohm–Gross response of the plasma, we express the source term (5.2) as a series expansion of derivatives of the delta function. It is easily shown that

$$S^{[\omega]}(\boldsymbol{r}, t) = \frac{q_o}{4\pi\varepsilon_o} \frac{\delta(r)}{r} \frac{\omega_p}{\omega_o^2 - \omega_p^2} \left[\frac{\omega_o}{\omega_p} \sum_{k=0}^{\infty} \frac{(-1)^k}{\omega_p^{2k}} \delta^{(2k)}(t) - \frac{\omega_p}{\omega_o} \sum_{k=0}^{\infty} \frac{(-1)^k}{\omega_o^{2k}} \delta^{(2k)}(t) \right]. \tag{5.4}$$

Further, the space and time evolution of the solution of the propagation equation (2.4) may be formally deduced from the formula

$$\Psi^{[\omega]}(r,t) = \frac{\omega_p}{\omega_o^2 - \omega_p^2}[\mathcal{C}_1(r,t) - \mathcal{C}_2(r,t)] \tag{5.5}$$

where the following abbreviations have been introduced:

$$\mathcal{C}_1(r,t) = \frac{\omega_o}{\omega_p}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_p^{2k}}\frac{\partial^{2k}h(r,t)}{\partial t^{2k}} \tag{5.6}$$

$$\mathcal{C}_2(r,t) = \frac{\omega_p}{\omega_o}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_o^{2k}}\frac{\partial^{2k}h(r,t)}{\partial t^{2k}}. \tag{5.7}$$

We recall that the function $h(r,t)$, as it is defined in (4.6) is nothing but the first derivative in r of the fundamental solution, $g_0(r,t)$, of the Klein–Gordon equation. Up to now, no condition on the value of the exciting frequency as compared with the plasma frequency has been required. It is easy to see that for the convergence of the series in (5.7) and for the validity of the present method, the necessary condition is $\omega_o \geqslant \omega_p$.

Since the algebraic expression of the function $\mathcal{C}_1(r,t)$ in (5.6) may be straightforwardly determined using the same procedure as in section 4 (see equation (4.5)), we have

$$\mathcal{C}_1(r,t) = 2\omega_o H(t - r/c_T)U_1(\omega_p(t - r/c_T), \omega_p(t^2 - r^2/c_T^2)^{1/2}). \tag{5.8}$$

Directing our attention to the second function $\mathcal{C}_2\,(r,t)$ in (5.7) this term becomes, similar to (4.12),

$$\mathcal{C}_2(r,t) = \mathcal{F}'(r,t) - H(t - r/c_T)\{\mathcal{D}_1'(r)\cos[\omega_o(t - r/c_T)] + \mathcal{D}_2'(r)\sin[\omega_o(t - r/c_T)]\} \tag{5.9}$$

with

$$\mathcal{F}'(r,t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_o^{2k}}\left[\frac{\partial^{2k}h_1}{\partial t^{2k}}\right]. \tag{5.10}$$

$\mathcal{D}_1'(r) = \delta(t - r/c_T)\mathcal{F}'(r,t)$, and $\mathcal{D}_2'(r) = -\omega_o + \omega_o^{-1}\delta(t - r/c_T)[\partial\mathcal{F}'(r,t)/\partial t]$. The function $h_1(r,t)$ has been defined in (4.7). With the help of the integral representation (4.16), we imitate the procedure used in the above section to express (5.10) as an integral. That is,

$$\mathcal{F}'(r,t) = -\omega_p H(t - r/c_T)\frac{1}{\pi}\int_0^{\pi} e^{\omega_p(r/c_T)\cos\zeta}\cos\zeta\cos(\omega_p t\sin\zeta) \times \sum_{k=0}^{\infty}((\omega_p/\omega_o)^2\sin^2\zeta)^k\,d\zeta. \tag{5.11}$$

Removing the series outside the integration operation, and introducing the parameter, $\beta = \cosh^{-1}(\omega_o/\omega_p)$, we have

$$\sum_{k=0}^{\infty}((\omega_p/\omega_o)^2\sin^2\zeta)^k = \frac{4e^{-2\beta}\cosh^2\beta}{1 + 2e^{-2\beta}\cos 2\zeta + e^{-4\beta}}. \tag{5.12}$$

The function goes back to a series representation if we use the formula (see [18] p 94 formula 501)

$$\frac{\cos\zeta}{1 - 2\kappa\cos 2\zeta + \kappa^2} = \frac{1}{1-\kappa}\sum_{n=0}^{\infty}\kappa^n\cos[(2n+1)\zeta] \tag{5.13}$$

with the understanding that $\kappa = -\exp(-2\beta) < 1$. Equation (5.11) thus takes the equivalent form given by

$$\mathcal{F}'(r,t) = -\omega_p H(t - r/c_T)\sum_{n=0}^{\infty}(-1)^n \mathrm{e}^{-2n\beta}\,\frac{1+\mathrm{e}^{-2\beta}}{2} \times\left[\frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{\omega_p(r/c_T)\cos\zeta}\cos[\omega_p t\sin\zeta + (2n+1)\zeta]\,\mathrm{d}\zeta + \frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{\omega_p(r/c_T)\cos\zeta}\cos[\omega_p t\sin\zeta - (2n+1)\zeta]\,\mathrm{d}\zeta\right]. \tag{5.14}$$

The transformation of $\mathcal{F}'(r,t)$ in terms of Lommel functions readily follows with the aid of Schläfli's integral formula (4.16). On the other hand, we can show that, $\mathcal{D}_1'(r) = -\omega_o\sin(\mathrm{e}^{-\beta}\omega_p(r/c_T))$, and $\mathcal{D}_2'(r) = -\omega_o\cos(\mathrm{e}^{-\beta}\omega_p(r/c_T))$. After some elementary transformations and algebra, the above results lead to the potential given by

$$\phi_{BG}^{[\omega]}(r,t) = \frac{q_o}{4\pi\varepsilon_o r}\frac{\omega_p}{\omega_o^2 - \omega_p^2}[\mathcal{C}_1(r,t) - \mathcal{C}_2(r,t)] \tag{5.15}$$

where $\mathcal{C}_1(r,t)$ has been defined in (5.8) and $\mathcal{C}_2(r,t)$ is expressed in terms of Lommel functions of two real variables as well. The solution consists of the particular limit when the collision may be neglected in the general result given by Randriamboarison [21]. This result may be put in a more suitable representation in which the steady-state term is separated from the transients. That is,

$$\mathcal{C}_2(r,t) = \omega_p H(t - r/c_T)[\sin(\omega_o t - k_o r) - \Upsilon(r,t)] \tag{5.16}$$

where, $k_o = c_T^{-1}(\omega_o^2 - \omega_p^2)^{1/2}$, stands for the modulus of the wavevector deduced from the Bohm–Gross dispersion relation, and

$$\Upsilon(r,t) = U_1(\omega_p\mathrm{e}^{-\beta}(t + r/c_T), \omega_p(t^2 - r^2/c_T^2)^{1/2}) - U_1(\omega_p\mathrm{e}^{-\beta}(t - r/c_T), \omega_p(t^2 - r^2/c_T^2)^{1/2}). \tag{5.17}$$

The function $\mathcal{C}_1(r,t)$ is a solution of the homogeneous linear equation associated with (2.4) subject to the initial conditions imposed by the problem. One can obviously show that, when added to the second term on the right-hand side of (5.3), this solution vanishes as t tends to infinity, and defines a transient response of the plasma. On the other hand, the component term which involves the $\Upsilon(r,t)$ is important only during the earlier stage of the time evolution of the signal, and it settles down to a negligible value when $t \gg 1$. The availability of the exact form of these transient responses is of essential interest in some physical applications. Their time decay may indeed be used to state the validity condition for the classical harmonic approximation description of the wave excitation (see [21]). As an example, we present in figure 2 the space-time variation of the envelope of the normalized function, $\omega_p^{-1}\mathcal{C}_2(r,t)$, which may be regarded as the Bohm–Gross wave density during its propagation. The wave is excited by an oscillatory charge at the frequency $(\omega_o/\omega_p) = 1.2$. The steady-state approximation is valid if the magnitude of the ripples observed at the plotted surface may be neglected. It is apparent in figure 2 that this condition depends on the distance from the transmitter and the observation point.

6. Response to a continuous excitation in a dense plasma

The power of the present formalism is not completely demonstrated without showing its effectiveness in the explicit display of the overdense plasma excitation. However, following the same procedure as in section 5, a complication arises in the case where $\omega_o < \omega_p$. Indeed, at

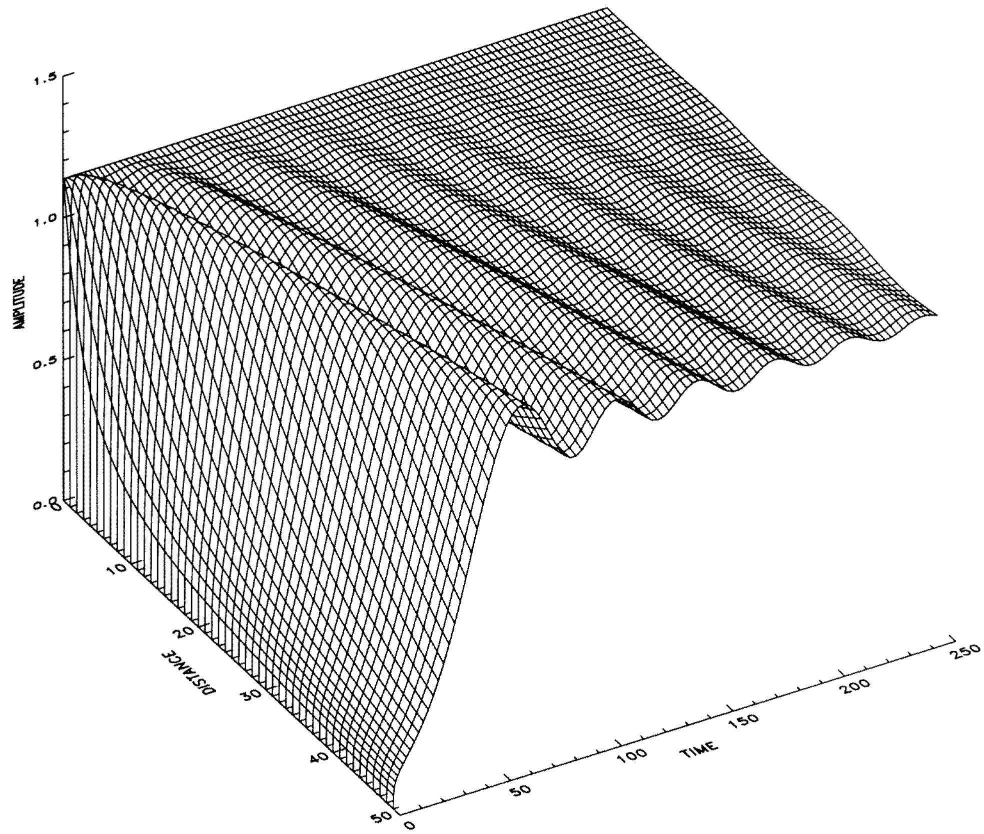

Figure 2. Spatio-temporal evolution of the envelope curve of the function ${\omega_p}^{-1}\mathcal{C}_2(r,t)$ (see (5.16)). At ultimate time, this function describes the dynamics of the steady-state Bohm–Gross wave. Here, the excitation consists on a pulsatory external charge at a frequency such as $(\omega_o/\omega_p) = 1.2$.

first sight, the convergence of the infinite series under the integral sign of (5.11) seems *a priori* to be not guaranteed. We can circumvent such an obstacle by use of the following approach.

We recall the notation of (4.7), $h_1(r,t) = [\partial g_0/\partial r]$. This function may be put in the form

$$h_1(r,t) = -2\omega_p H(t - r/c_T)\frac{1}{\pi}\int_0^{\pi/2} \sinh[\omega_p(r/c_T)\cos\zeta]\cos\zeta\cos(\omega_p t\sin\zeta)\,\mathrm{d}\zeta \qquad (6.1)$$

where the integration now runs from 0 to $\frac{\pi}{2}$. Equation (6.1) is then split into the sum of two integrals. We set $h_1(r,t) = h_1^{[1]}(r,t) + h_1^{[2]}(r,t)$; the component terms are obtained by decomposing the interval of integration into $[0, (\frac{\pi}{2}-\alpha)]$ and $[(\frac{\pi}{2}-\alpha), \frac{\pi}{2}]$, respectively, where

$$\alpha = \cos^{-1}(\omega_o/\omega_p). \qquad (6.2)$$

According to (5.4), we now write

$$\mathcal{C}_2(r,t) = \frac{\omega_p}{\omega_o}\{\omega_o H(t - r/c_T)\sin[\omega_o(t - r/c_T)] + \mathcal{H}_1(r,t) + \mathcal{H}_2(r,t)\} \qquad (6.3)$$

with the notations

$$\mathcal{H}_1(r,t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega_o^{2k}}\left(\frac{\partial^{2k}h_1^{[1]}}{\partial t^{2k}}\right) \qquad (6.4)$$

and

$$\mathcal{H}_2(r,t) = \omega_o H(t)\sin(\omega_o t) *_t h_1^{[2]}(r,t). \tag{6.5}$$

Proceeding globally along the same lines as in section 5, the function in (6.4) may be put in the form

$$\mathcal{H}_1(r,t) = \mathcal{F}_1(r,t) - H(t - r/c_T) \times\{\mathcal{A}_1^{[1]}(r)\cos[\omega_o(t - r/c_T)] + \mathcal{A}_2^{[1]}(r)\sin[\omega_o(t - r/c_T)]\} \tag{6.6}$$

where $\mathcal{F}_1(r,t)$ is expressible as

$$\mathcal{F}_1(r,t) = -4\frac{\omega_p^2}{\omega_o^2}\cos^2\alpha\, H(t - r/c_T)\int_0^{\pi/2-\alpha} u(r,t,\zeta)\,\mathrm{d}\zeta \tag{6.7}$$

the integrand of which is the rational fraction of trigonometric functions:

$$u(r,t,\zeta) = \frac{1}{\pi}\sinh[\omega_p(r/c_T)\cos\zeta]\frac{\cos\zeta\cos(\omega_p t\sin\zeta)}{\cos 2\alpha + \cos 2\zeta}. \tag{6.8}$$

Moreover, in (6.6), we have used the notations $\mathcal{A}_1^{[1]}(r) = \delta(t - r/c_T)\mathcal{F}_1(r,t)$ and $\mathcal{A}_2^{[1]}(r) = \omega_o^{-1}\delta(t - r/c_T)[\partial\mathcal{F}_1(r,t)/\partial t]$. The function $\mathcal{F}_1(r,t)$ is the sum of all successive odd derivatives of $h_1^{[1]}(r,t)$ without considering its discontinuity, and the remaining two terms on the right-hand side of (6.6) constitute the contribution of the discontinuity at $t = (r/c_T)$.

On the other hand, the corresponding integrals associated with the term $\mathcal{H}_2(r,t)$ in (6.5) may be displayed if we expand the source term as a power series of t using Taylor expansion or Borel's representation. That is,

$$\sin(\omega_o t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\omega_o^{2k+1}t^{2k+1}. \tag{6.9}$$

We make use of Dirichlet's integral formula (3.7) to express the time convolution of each term of this series and the function $h_1^{[2]}(r,t)$. That is, for $n = 2(k+1)$, $k \geqslant 0$, we define the function $\mathcal{I}_k(r,t)$ as

$$\begin{aligned}\mathcal{I}_k(r,t) &= \frac{1}{(2k+1)!}\int_0^t (t - t_1)^{2k+1} h_1^{[2]}(r,t_1)\,\mathrm{d}t_1\\ &= \int_{r/c_T}^{t}\int_{r/c_T}^{t_{2(k+1)}}\cdots\int_{r/c_T}^{t_2} h_1^{[2]}(r,t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2\ldots\mathrm{d}t_{2(k+1)}\\ &= \int_{r/c_T}^{t}\cdots\int_{r/c_T}^{t} h_1^{[2]}(r,t)(\mathrm{d}t)^{2(k+1)}.\end{aligned} \tag{6.10}$$

It has been assumed in (6.10) that $h_1^{[2]}(r,t) \equiv 0$ for $t < (r/c_T)$. In section 5, it was seen that the discontinuity of the fundamental solution contributes significantly to the complete expression of the plasma response. An explicit elucidation of higher-order time integrals of the function $h_1^{[2]}(r,t)$, considering its simple jump at the point $t = (r/c_T)$ has been performed in appendix B. Upon integrating $h_1^{[2]}(r,t)$ with respect to the variable t, the tricky factor, $(\omega_p\sin\zeta)$ is now injected into the denominator of the integrand of, respectively, $\mathcal{J}_k(r,t)$, $\mathcal{J}_k'(r,t)$ and $\mathcal{J}_k''(r,t)$ (see appendix B, equations (B.6)–(B.8)). When summing the resulting expressions according to the definition of equation (6.4), we can employ a suitable form for $\mathcal{H}_2(r,t)$. The result turns out to be

$$\mathcal{H}_2(r,t) = \mathcal{F}_2(r,t) - H(t - r/c_T) \times\{\mathcal{A}_1^{[2]}(r)\cos[\omega_o(t - r/c_T)] + \mathcal{A}_2^{[2]}(r)\sin[\omega_o(t - r/c_T)]\} \tag{6.11}$$

with

$$\mathcal{F}_2(r,t) = -4\frac{\omega_p^2}{\omega_o^2}\cos^2\alpha\, H(t - r/c_T)\int_{\pi/2-\alpha}^{\pi/2} u(r,t,\zeta)\,\mathrm{d}\zeta.$$

$\mathcal{A}_1^{[2]}(r) = \delta(t - r/c_T)\mathcal{F}_2(r,t)$, and $\mathcal{A}_2^{[2]}(r) = \omega_o^{-1}\delta(t - r/c_T)[\partial\mathcal{F}_2(r,t)/\partial t]$. We observe that the integrand of $\mathcal{F}_2(r,t)$ has the identical form as that of $\mathcal{F}_1(r,t)$ defined in (6.8). The convergence of the initial series is now proved, and we are in a position to obtain the entire integral form of the function $\mathcal{C}_2(r,t)$ expressed in (6.3). Summing (6.6) and (6.11) we obtain

$$\mathcal{C}_2(r,t) = \mathcal{F}''(r,t) - H(t - r/c_T)\{\mathcal{D}_1''(r)\cos[\omega_o(t - r/c_T)] + \mathcal{D}_2''(r)\sin[\omega_o(t - r/c_T)]\} \tag{6.12}$$

where

$$\mathcal{F}''(r,t) = -2\frac{\omega_o}{\omega_p}H(t - r/c_T)\frac{1}{\pi}\int_0^{\pi/2} \mathrm{e}^{\omega_p(r/c_T)\cos\zeta}\,\frac{\cos\zeta\,\cos(\omega_p t\sin\zeta)}{\cos 2\alpha + \cos 2\zeta}\,\mathrm{d}\zeta.$$

$\mathcal{D}_1''(r) = \mathcal{A}_1^{[1]}(r) + \mathcal{A}_1^{[2]}(r)$, and $\mathcal{D}_2''(r) = -\omega_o + \mathcal{A}_2^{[1]}(r) + \mathcal{A}_2^{[2]}(r)$. Hence, both the leading term and contributions of the jump in (6.12) are now amendable along the same lines as that developed in the previous two sections, i.e. sections 4 and 5. The excited thermal plasma wave component, $\phi_{GB}^{[\omega]}(\boldsymbol{r},t)$, is expressed as the sum of two terms as in (5.15), but, instead of (5.16), the second term of the formula takes the form [21] of

$$\mathcal{C}_2(r,t) = -2\omega_p H(t - r/c_T)\,\mathrm{Im}\,\Lambda_1(\mathrm{i}\omega_p\mathrm{e}^{\mathrm{i}\alpha}(t - r/c_T), \omega_p(t^2 - r^2/c_T^2)^{1/2}). \tag{6.13}$$

The symbol Im denotes here the imaginary part of the complex quantity throughout, and the notation $\Lambda_n(w,z)$ represents the nth-order modified two-variable Lommel functions of the first kind [22]. Here, the first argument w of these functions is complex.

It should be emphasized that there is a great variety of special function solutions of linear differential equations such as the Klein–Gordon equation. If these solutions are too complicated, their physical significance is not understood and they stand only as mathematical curiosities. For the special functions considered in (6.13), mathematical properties and numerical evaluation are now available [21, 22], so insight into physical behaviour of the solution may be acquired without difficulty, in contrast to the classical analytical responses which involve integrals [23, 24, 25]. An exact and algebraic representation of the plasma response at the resonant excitation, i.e., the solution when the excitation frequency equals the plasma frequency, may be deduced as a limit of the above solution. Then, not only does the approach give the behaviour of the secular oscillations of the plasma, but it also clarifies and highlights previous results on the radiation processes [21]. As an example of numerical result, plotted in figure 3 is the space-time variation of the amplitude of an evanescent Bohm–Gross wave. Note that description and analysis of the evolution of such a wave are given in [19].

7. Summary and concluding remarks

The response of an isotropic warm plasma subject to an external excitation may be represented as the summation of two components as, $\phi(r,t) = \phi_C(r,t) + \phi_{BG}(r,t)$. The first term corresponds to the potential that would exist in a cold plasma, while the second represents a warm plasma wave field. The time evolution of the cold plasma response, although not usually found in textbooks on plasma physics, is elementary and well known. It turns out to be the sum of the external potential expressed by equation (4.1) or (5.1) and the collective electron oscillations given by equation (4.3) or (5.3) for the impulsive or periodic temporal excitation.

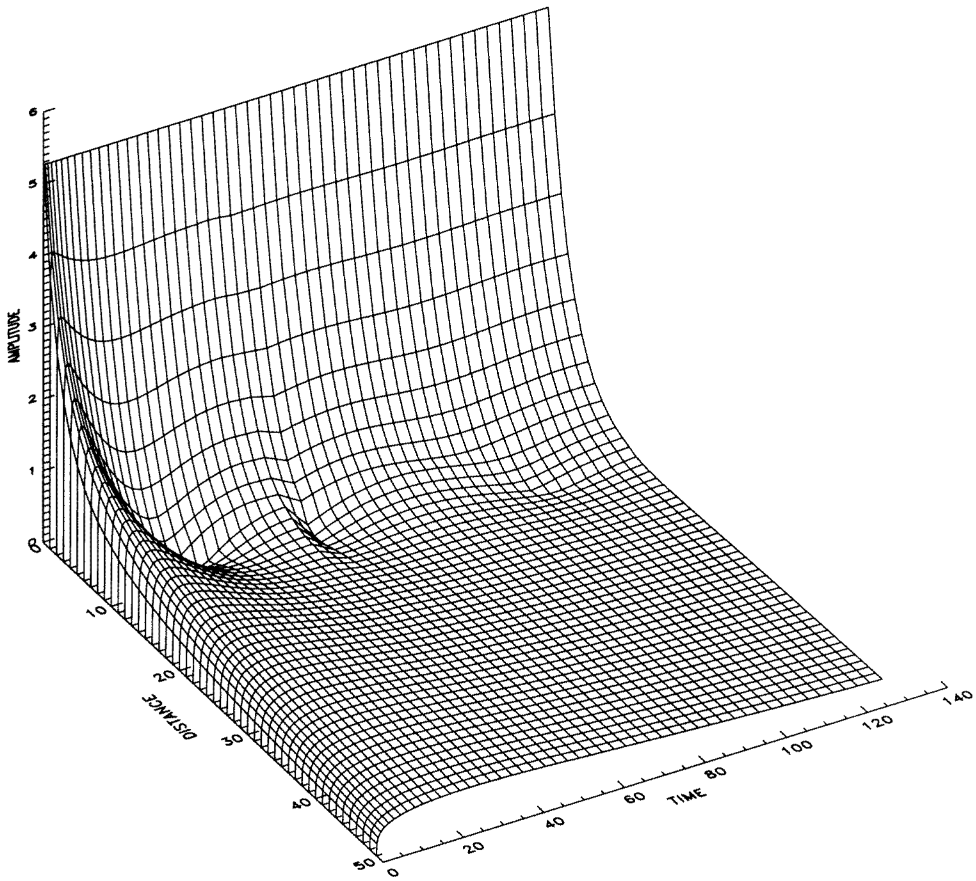

Figure 3. As in figure 2, but $(\omega_o/\omega_p) = 0.9$. The evolution of the evanescent Bohm–Gross wave is described.

On the other hand, the second response is known to be a solution of a non-homogeneous linear Klein–Gordon equation. To deal with such an equation, an alternative approach based on the theory of distributions (in the sense of L Schawrtz), associated with some mathematical transforms and initiated by Robinson [1] has been proposed in this paper. Because of the linear nature of the problem, the superposition principle applies and the resulting solution involves a convolution integral of the Klein–Gordon fundamental solution and the source term. If the source term can be expanded in terms of its moments via Poisson's transform, it has been shown that $\phi_{BG}(r, t)$ is an infinite series of the derivatives of the fundamental Klein–Gordon solution $g_0(r, t)$ defined in (3.5). A detailed description of the procedure has been presented. Several important aspects of the theory that were not addressed in the original formulation of Robinson [1] were revealed as a consequence. In particular, the effects of the simple discontinuity of Green's generalized function have been fully investigated.

The development exhibited above undoubtedly holds for a class of source term functions of space and time. The physically motivated problems considered in this paper suggested, however, a broader scope of the method. Expanding the driving force of the radiation equation by means of Borel's transform (closely related to Taylor development), it was shown that the solution may be represented as a finite or infinite series of repeated integrals of the Green function $g_0(r, t)$.

As a consequence, the exact algebraic expression of the impulsive response of the plasma [2] was recovered by an approach different from the standard Laplace transform scheme. The wave potential is given in terms of a Lommel function of two real variables defined in (4.21). In addition, the equivalent exact representation of the longitudinal Bohm–Gross harmonic wave, which agrees with the previous and more general result in [21], is also inferred. In this case, equation (5.5) underlines the global form of the wave potential $\phi_{BG}(r, t)$. Two distinct components make up the solution. The first term, i.e. (5.8), involves a transient part of the response. In the case of the propagating and dispersive wave situation ($\omega_p < \omega_0$), the second term on the right-hand side of (5.5) takes the form of (5.16). In the case of the evanescent wave, this second component takes the form of (6.13) and exhibits a modified Lommel function.

The excited medium under investigation is assumed to be an infinite fluid unmagnetized electron plasma. Nevertheless, the subject presents some interesting aspects, in particular on the improvement of the theoretical model for the response of active sounders in laboratory and space plasma science. The solution of elementary examples, as undertaken here, is intended to provide a theoretical basis for treatment of more complicated but realistic problems related to experimental situations.

The work is applicable to a variety of wave propagation phenomena of plasma physics, including electromagnetic waves, acoustic waves, etc. For instance, the cold plasma electromagnetic wave expressed by Felsen and Marcuvitz [26], or by Borisov and Simonenko [27] in terms of integrals of Bessel functions may be written and analysed more efficiently by means of Lommel special functions. Similarly, an alternative formulation of the problem of the ultra-wideband electromagnetic pulse propagation in the ionosphere [28] can also be effected.

Acknowledgments

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Appendix A. On the Green function of the Klein–Gordon equation

In this appendix, an outline of the demonstration of equations (3.3)–(3.5), initially pointed out in Robinson's note [1], is presented.

The causal Green function is the solution of the particular excitation problem governed by the inhomogeneous equation

$$\mathcal{L}g(r, t) \equiv \left(-\frac{\partial^2}{\partial r^2} + \frac{1}{c_T^2}\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{c_T^2} \right) g(r, t) = \delta(r)\delta(t) \tag{A.1}$$

with the prescribed Cauchy data, $g(r, t \leqslant 0) = 0$, $(\partial/\partial t)g(r, t \leqslant 0) = 0$.

In the sense of distribution, equation (A.1) can be differentiated j times with regard to r, and k times with regard to t. Using the commutative property of partial derivatives, one can show that the function $\varphi(r, t) = (\partial^{j+k} g(r, t)/\partial r^j \partial t^k)$ satisfies the relation

$$\mathcal{L}\varphi(r, t) = \delta^{(j)}(r)\delta^{(k)}(t). \tag{A.2}$$

A simple illustration of the relevance of the above formalism may be glimpsed in the following development.

Morse and Feshbach [29] discussed the Green function of the Klein–Gordon equation in three-dimensional space. They started with the equation of the form

$$\nabla^2 G(\boldsymbol{r}, t) - \frac{1}{c_T^2}\frac{\partial^2}{\partial t^2}G(\boldsymbol{r}, t) - \frac{\omega_p^2}{c_T^2}G(\boldsymbol{r}, t) = -4\pi\delta_3(\boldsymbol{r})\delta(t) \tag{A.3}$$

and found that its solution is given by

$$G(\boldsymbol{r}, t) \equiv G(r, t) = \frac{1}{r}\frac{\partial k(r, t)}{\partial r} \tag{A.4}$$

with

$$k(r, t) = -c_T H(t - r/c_T) J_0[\omega_p (t^2 - r^2/c_T^2)^{1/2}]. \tag{A.5}$$

As $k(r, t)$ is nothing but the function $g_0(r, t)$ introduced in (3.5), our goal consists therefore in demonstrating the equality between this function and the solution of (A.1).

As far as isotropic and spherically symmetric problems are concerned, we may write

$$\begin{aligned}\nabla^2 G(\boldsymbol{r}, t) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) G(r, t) \\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) G(r, t)\end{aligned} \tag{A.6}$$

and

$$\delta_3(\boldsymbol{r}) = \frac{1}{4\pi r^2}\delta(r). \tag{A.7}$$

Substituting (A.6) and (A.7) in (A.3), it is easy to check that the resulting Green equation is equivalent to

$$\mathcal{L}[rG](r, t) = \delta'(r)\delta(t). \tag{A.8}$$

The relation $r^{-1}\delta(r) = -\delta'(r)$ has been used when establishing (A.8). Now, let $\Gamma(g, rG)$ be the combination of convolution products defined by

$$\Gamma(g, rG) = g(r, t) * \mathcal{L}[rG](r, t)] - [rG](r, t) * \mathcal{L}g(r, t). \tag{A.9}$$

According to equations (A.1) and (A.8), equation (A.9) may be rewritten as

$$\begin{aligned}\Gamma(g, rG) &= g(r, t) * \delta'(r)\delta(t) - [rG](r, t) * \delta(r)\delta(t) \\ &= \frac{\partial g(r, t)}{\partial r} - rG(r, t).\end{aligned} \tag{A.10}$$

Furthermore, if we make use of Green's theorem ([17] p 171, for example), we find that $\Gamma(g, rG)$ is expressible in the form

$$\begin{aligned}\Gamma(g, rG) = &\int_\Delta c_T^{-1}\left[g(r - r_1, t - t_1)\frac{\partial [rG](r_1, t_1)}{\partial t} - [rG](r - r_1, t - t_1)\frac{\partial g(r_1, t_1)}{\partial t}\right]_0^{t+0} \mathrm{d}r_1 \\ &+ \int_0^t \mathrm{d}t_1 \int_S \left(g(r - r_1, t - t_1)\frac{\partial [rG](r_1, t_1)}{\partial r}\right. \\ &\left. - [rG](r - r_1, t - t_1)\frac{\partial g(r_1, t_1)}{\partial r}\right) \mathrm{d}S\end{aligned} \tag{A.11}$$

where S corresponds to the bounding surface of a spherical space domain Δ of radius R ($R \to +\infty$), and $t + 0 = \lim_{\epsilon\to 0}(t + \epsilon)$, ($\epsilon \geqslant 0$). Using the physically acceptable initial and boundary value conditions assumed for the solutions g and G, it is a trivial matter to show that the right-hand side of (A.11) vanishes. Consequently, we have the relation

$$G(r, t) = \frac{1}{r}\frac{\partial g(r, t)}{\partial r} \tag{A.12}$$

and, in so far as evolutionary responses which tend to zero at $r \to +\infty$ are involved, this establishes the identity of $k(r, t)$ in (A.5) and the researched solution $g(r, t)$ of (A.1).

Appendix B. Multiple iterated integral of the Green function

The following method has proved useful in order to evaluate the integral

$$\mathcal{I}_k(r,t) = \int_{r/c_T}^{t} \cdots \int_{r/c_T}^{t} h_1^{[2]}(r,t)(\mathrm{d}t)^{2(k+1)} \tag{B.1}$$

where

$$h_1^{[2]}(r,t) = -2\omega_p H(t - r/c_T)\frac{1}{\pi}\int_{\pi/2-\alpha}^{\pi/2} \sinh[\omega_p(r/c_T)\cos\zeta]\cos\zeta\,\cos(\omega_p t\sin\zeta)\,\mathrm{d}\zeta.$$

We start by noting that, $(\mathrm{d}^n/\mathrm{d}t^n)\cos(\Omega t) = \Omega^n\cos(\Omega t + n\pi/2)$. Although the relation is valid for all integer n, our particular interest here is in $n = 2(k+1)$, $k \geqslant 0$, and $\Omega = \omega_p \sin\zeta$, for which we can write

$$\cos(\omega_p t\sin\zeta) = \frac{(-1)^{k+1}}{\omega_p^{2(k+1)}\sin^{2(k+1)}\zeta}\frac{\mathrm{d}^{2(k+1)}}{\mathrm{d}t^{2(k+1)}}(\cos(\omega_p t\sin\zeta)). \tag{B.2}$$

Upon integrating equation (B.2) $2(k+1)$ times, we arrive at

$$\int_{r/c_T}^{t} \cdots \int_{r/c_T}^{t} \cos(\omega_p t\sin\zeta)(\mathrm{d}t)^{2(k+1)} = \frac{(-1)^{k+1}}{\omega_p^{2(k+1)}\sin^{2(k+1)}\zeta}\int_{r/c_T}^{t} \cdots \int_{r/c_T}^{t} \frac{\mathrm{d}^{2(k+1)}}{\mathrm{d}t^{2(k+1)}}\cos(\omega_p t\sin\zeta)(\mathrm{d}t)^{2(k+1)}. \tag{B.3}$$

In the context of the derivation of the Taylor expansion of a function $f(x)$ of a real variable x about a regular point a, the Lagrange remainder of the series is established [30] as

$$\int_a^x \cdots \int_a^x f^{(n)}(x)(\mathrm{d}x)^n = f(x) - \sum_{j=0}^{n-1}\frac{(x-a)^j}{j!}f^{(j)}(a) \tag{B.4}$$

where $f^{(n)}$ represents the nth-order derivative of the f. Using the formula (B.4), the right-hand side of the equation (B.3) can be transformed and we have

$$\begin{aligned}\int_{r/c_T}^{t} \cdots \int_{r/c_T}^{t} \cos(\omega_p t\sin\zeta)(\mathrm{d}t)^{2(k+1)} &= \frac{(-1)^{k+1}}{\omega_p^{2(k+1)}\sin^{2(k+1)}\zeta}\Bigg\{\cos(\omega_p t\sin\zeta) \\ &\quad - \sum_{j=0}^{2k+1}\frac{(t-r/c_T)^j}{j!}\left(\frac{\mathrm{d}^j}{\mathrm{d}t^j}\cos(\omega_p t\sin\zeta)\right)_{t=r/c_T}\Bigg\} \\ &= \frac{(-1)^{k+1}}{\omega_p^{2(k+1)}\sin^{2(k+1)}\zeta}\Bigg\{\cos(\omega_p t\sin\zeta) \\ &\quad + \sum_{l=0}^{k}(-1)^l\frac{\omega_p^{2l+1}\sin^{2l+1}\zeta}{(2l+1)!}(t-r/c_T)^{2l+1}\sin[\omega_p(r/c_T)\sin\zeta] \\ &\quad - \sum_{l=0}^{k}(-1)^l\frac{\omega_p^{2l}\sin^{2l}\zeta}{(2l)!}(t-r/c_T)^{2l}\cos[\omega_p(r/c_T)\sin\zeta]\Bigg\}.\end{aligned}$$

Making use of the definition of the function $h_1^{[2]}(r,t)$, the integral $\mathcal{I}_k(r,t)$ may be expressed as the sum

$$\mathcal{I}_k(r,t) = \mathcal{J}_k(r,t) + \mathcal{J}_k'(r,t) + \mathcal{J}_k''(r,t) \tag{B.5}$$

where

$$\mathcal{J}_k(r,t) = 2\frac{(-1)^k}{\omega_p^{2k+1}} H(t - r/c_T)\frac{1}{\pi}\int_{\pi/2-\alpha}^{\pi/2} \sinh[\omega_p(r/c_T)\cos\zeta]\frac{\cos\zeta\,\cos(\omega_p t\sin\zeta)}{\sin^{2(k+1)}\zeta}\,\mathrm{d}\zeta \quad \text{(B.6)}$$

$$\mathcal{J}_k'(r,t) = 2\frac{(-1)^k}{\omega_p^{2k+1}} H(t - r/c_T)\sum_{l=0}^{k}(-1)^l\frac{\omega_p^{2l+1}}{(2l+1)!}(t - r/c_T)^{2l+1}$$
$$\times\frac{1}{\pi}\int_{\pi/2-\alpha}^{\pi/2} \sinh[\omega_p(r/c_T)\cos\zeta]\frac{\cos\zeta\,\sin[\omega_p(r/c_T)\sin\zeta]}{\sin^{2(k-l)}\zeta}\,\mathrm{d}\zeta \quad \text{(B.7)}$$

and

$$\mathcal{J}_k''(r,t) = -2\frac{(-1)^k}{\omega_p^{2k+1}} H(t - r/c_T)\sum_{l=0}^{k}(-1)^l\frac{\omega_p^{2l}}{(2l)!}(t - r/c_T)^{2l}$$
$$\times\frac{1}{\pi}\int_{\pi/2-\alpha}^{\pi/2} \sinh[\omega_p(r/c_T)\cos\zeta]\frac{\cos\zeta\,\cos[\omega_p(r/c_T)\sin\zeta]}{\sin^{2(k-l+1)}\zeta}\,\mathrm{d}\zeta. \quad \text{(B.8)}$$

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